Fibonacci

The Man And His Work

An Essay by

SCOTT WILSON

Contents

Introduction:

Fibonacci was one of the earliest of the great modern European mathematicians. He is best known for the sequence of numbers that bears his name, but he made many useful contributions to mathematics, and this is the story of his life and achievements.

Note: Numerical references refer to the numbered entries in the Bibliography. Symbolic references refer to footnotes on the same page.

Who He Was:

Leonardo Pisano ('of Pisa') was born sometime around 1170-1180 in Pisa, central Italy. In the late 12th century, Pisa was an independent city-state, with a population of around 10,000.[20] His father, Gulielmo Bonacci, was a merchant,[20] scribe,[19] customs manager[10] or administrator[1] (depending on the source) for the Pisan government, and the name 'Fibonacci' means 'son of Bonacci' or 'of the Bonacci family'. The surname 'Bonacci' on the father is not absolutely certain, and the definition 'fils du bonace' ('son of a good fellow') as the origin of the word Fibonacci has been suggested. The name was first used by Guillaume Libri,[8] a 19th century historian, who may also have been the first to apply the name to the sequence.

When his father was stationed in Bougie (Bugia) in Algeria, Northern Africa, the young Leonardo went with him. There he received an early introduction to the Hindu-Arabic numerals and calculation methods, and he was taught by a Muslim teacher.[5] Seeing numbers being used daily in his father's business transactions, Leonardo gained an early interest in mathematics. As he became independent, Leonardo travelled widely around the Mediterranean, to Arabian ports, Egypt, Sicily, Greece, Syria, Constantinople, France, and of course Italy. Wherever he went, Leonardo took careful notice of the arithmetic and commerce systems in use, and visited and learned from as many local scholars as he could. Not a lot is known about his later life, other than he spent much of it in Pisa writing mathematical books, corresponding with various other people, and doing mathematics. He survived by commerce and accounting, and in 1240, he was honoured by the city of Pisa for his accounting and other services to the state, and was awarded an annual pension.[19] This information was recorded on a marble tablet at the time. It is not known whether he ever married or fathered children. He died around 1250, possibly but not definitely in Pisa, as Pisa was defeated in a naval battle with Genoa around that time.[20]

1: Pisa   2: Bougia   3: Palermo   4: Constantinople

What He Wrote

Leonardo returned to Pisa in about 1200, and there began to write his mathematical books. The first was *Liber Abaci*, ‘Book of the Abacus’/‘Book of Calculations’, which was finished around 1202 and was 15 chapters long, in 4 parts. It was also known as ‘Algebra et almuchabala’,[3] a title taken from a work named ‘Al-Kitāb al-mukhtasar fīhisāb al-jabr wa’l muqābala’ by al Khowârizmî,[20] who was one of the main influences for *Liber Abaci*. The influence of Abû Kâmil and Diophantus can also be seen.[15] One of its main subjects was the Hindu-Arabic number system, which Leonardo had found much better than the Roman numerals still being used in Europe at the time. One of the most important differences with the Roman numerals was the concept of a positional number system, and the associated idea of a ‘place holder’, or zero (known as a ‘zephirum’ from the Arabic ‘s̲ifr’ meaning ‘empty’, also the root of the word ‘cipher’). He gave detailed explanations on how to read, write and perform addition, subtraction, multiplication and division with the ‘new’ numerals. Fractions were explained, as was the use and calculation of square and cube roots. Leonardo used the Arabic style of putting the fractional part of a mixed fraction to the left of the whole part, as opposed to the modern way on the right. He also used a horizontal bar in his fractions, a practice that did not become mainstream until the 16th century.[5] The book also contained tables of unit fraction conversions for

common fractions, as Leonardo either preferred unit fractions himself, or believed that those reading his work did.[5] Modern style symbolic algebra was not available, and Leonardo wrote equations in a longhand text form, with 'radix' for √, 'quadratus' for x^2, and 'cubus' for x^3.[16] He also sometimes used 'res' to stand for an unknown quantity. These terms are translations of Arabic terms for the same quantities. Two methods for solving linear and quadratic problems are given, both false position and algebraic approaches. Leonardo ignored negative and imaginary roots to equations;[10] although negative values are considered in some later problems, particularly those involving finance to allow the case of being in debt. Several cases of division were given as well, using the 'scratch' or 'galley' method for long division, which was a difficult process in the days of Roman numerals.
Liber Abaci also contained applications of the new numbers to commerce and business transactions, such as bartering, partnership, alligation and pricing. The advantages of using geometry to solve algebraic problems are also shown in detail.
Liber Abaci included a large number of mathematical problems and riddles, and their solutions. These include the aforementioned business practices, often with each problem in as many variations as possible; related topics such as inheritance; geometry, arithmetic and geometric progressions including the rabbit-breeding problem that gives rise to the Fibonacci sequence; quadratic equations, problems involving square and cube roots, various other patterns such as 'perfect' numbers (the sum of the divisors other than the number itself is equal to the number) and 'friendly' numbers, and Diophantine equations. It also contained a version of the well known "As I was going to St. Ives..." riddle, although in this case it is presented as a serious problem in geometric series, and is about women, donkeys and bread. A similar problem, and calculated solution, was found on the Rhind papyrus (from 1650BC), involving wheat fields, but still the powers of 7.[10] When the problem was turned into a riddle with a twist is unknown. The *Liber Abaci* remained a standard text for over two hundred years.

Leonardo's next important work was the *Practica Geometriae,* finished in 1220. It was based on an Arabic version of Euclid's 'Division of Figures', and was also influenced by Hero's work on mensuration,[5] and 'Liber Embadorum' (1145) from Plato of Tivoli, which was in turn a translation of a treatise on areas in Hebrew by Savasorda.[15] The *Practica Geometriae* is mostly concerned with geometrical problems, some of which are solved using algebraic methods. It was fundamental to later studies of geometry. It also included problems and examples on measurement, with a proof of the formula to calculate the area of a triangle from the lengths of its sides, basically the same as Hero's formula.[*] Also given was a three-dimensional version of Pythagoras' Theorem. Trigonometric operations are dealt with, along with square and cube roots. Leonardo was also the first to realise that each square number is the sum of a series of odd numbers. *Practica Geometriae* also contained information on the use of the quadrans, a surveying instrument of the era.[16] Leonardo used 864/275 as an approximation to pi in his geometrical calculations, which is 3.1418…[9]

In 1225, Leonardo was invited to take part in a mathematical tournament at the court of the Holy Roman Emperor, Frederick II. The Emperor had read *Liber Abaci*, and wanted to see the mathematical skill of its author for himself. Three problems were set by Johannes of Palermo, a scholar in the Emperor's staff, and were sent out to the invitees in advance. Leonardo was presented to the Emperor by the astronomer Dominicus.[6] He gave correct answers to all three problems, while those selected to compete against him were stumped.

The three problems were:

1. Determine the values of x and y so $x^2 + 5 = y^2$ and $x^2 - 5 = z^2$.
2. Solve $x^3 + 2x^2 + 10x = 20$ for x.
3. Three men have shares of a half, a third and a sixth in an unknown amount of money. From this total, each man takes a random amount. Then each man returned, of what he held, a half, a third and a sixth, respectively. This returned amount was divided evenly to all three, which gave each man exactly what he was entitled to. How much does each man have?

(The first problems were given longhand, rather than algebraically).

Leonardo solved all three. The first has a fractional solution, $x = 41/12$, $y = 49/12$, and $z = 31/12$.[†]
The second problem could not be solved algebraically, and Leonardo proved why it has no rational solution and cannot be solved with a ruler and compass.[5] He then gave an approximate result in sexagesimal fractions, $1^0\ 22^1\ 7^2\ 42^3\ 33^4\ 4^5\ 40^6$, which is 1.3688081075, accurate to 10^{-10}. He gave no method for the calculation of this approximation.[16]
The solution to the third problem was contained in one of Leonardo's later books, *Flos*. The smallest whole solution is given as 47.[16]

---

[*] See section 'The Mathematics Involved', *Practica Geometriae*, page 9
[†] See section 'The Mathematics Involved', *Liber Quadratorum*, Proposition 17, page 16

Soon after the contest, Leonardo finished two books in 1225. The first was *Liber Quadratorum*, 'The Book of Squares', which was dedicated to the Emperor Frederick II. The introduction contained details of both Leonardo's meeting with the Emperor and the contest, particularly the first tournament problem. The book has 24 propositions, and is mostly concerned with problems involving squares and their solutions, but also involves determinate and indeterminate analysis, and makes use of the 'sum of odd numbers is a square' property. The tournament problem 1 is detailed in Proposition 17. The book contains the identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (bc - ad)^2$$
$$= (ad + bc)^2 + (ac - bd)^2$$

Also used in some of the propositions is the idea of 'congruous' numbers, used by Leonardo to descried numbers of the form '$ab(a+b)(a-b)$'. Proposition 12 shows that if a and b are relatively prime, the congruous number formed is a multiple of 24. For a more detailed examination of the *Liber Quadratorum*, see the section entitled 'The Mathematics Involved'.[13]

The other book written following the tournament was *Flos*, 'Flower' or 'Blossom', which was also known as *Flos Leonardi Bigollo Pisani super solutionibus quarundamquaestionum…*[19] Leonardo sometimes used the name 'Bigollo',[19] meaning 'fool', as many people were sceptical of the new numbers that he introduced, so he made fun of himself with the name; or meaning 'traveller', as he was widely travelled. The book was concerned mostly with cubic equations, and contained both of the other problems from the tournament, but still no method for his approximate solution of the cubic problem. It is thought that he may have used 'Horner's Method', an Arabian method that may have originally come from the Chinese.[5] Horner's method is similar to the Newton-Raphson method of finding the root of an equation used today. He also considered indeterminate problems with methods that had not been much used since Diophantus, and used Euclidean methodology and Arabian and Chinese techniques, learned in his earlier travels, to solve determinate problems.

He also wrote a letter to Theodorus (or Theodoris), entitled *Epsistola ad Magistrum Theodorum*,[13] who was a philosopher in Emperor Frederick II's court, soon after the tournament. In it, he again deals with the problems from the tournament. This is the only surviving letter written by Leonardo, and much of what is known about his life comes from it and the small biographies included in some of his manuscripts.

In 1228, Leonardo revised the *Liber Abaci*, and dedicated it to Michael Scot, a Scottish acquaintance of his, who was chief astrologer to the

Emperor and a writer of science texts.[16] It is this version that became widely distributed around Europe, and was instrumental in the widespread adoption of the Hindu-Arabic numerals.
The complete works of Leonardo were edited and published in 1862 by Baldassarre Boncompagni, under the title *Scritti di Leonardo Pisano*.[1] This may have been before the name 'Fibonacci' was given to him.

Leonardo also wrote a book on commercial arithmetic entitled *Di minor guisa*, and possibly a piece on Book X of Euclid's 'Elements'. Both are unfortunately lost.[13]

The Mathematics Involved

Leonardo performs his general analysis using labelled line segments as his variables, giving physical meaning to the adding of variables. It is at times very hard to follow, so I will only use modern notation when algebra is required.

Liber Abaci

The problems given in this book include a version of the riddle 'As I was going to St. Ives'. In Leonardo's case, there are seven old women going to Rome, each bringing seven mules. Each mule carries seven sacks, and each sack contains seven loaves of bread. Accompanying each loaf is seven knives, and to protect each knife, there are seven sheaths. The total is shown to come to $7 + 7^2 + 7^3 + 7^4 + 7^5 + 7^6 = 137256$.
Leonardo also looks at problems involving inheritance, such as:
A dying man gathers his sons around him to split up his wealth. To his first son, he gives one bezant and 1/7 of the remaining stockpile. To his second son, he gives two bezants and 1/7 of the remainder. To the third, he leaves three bezants and 1/7 of what's left. This continues in the same fashion until the last son, who he gives the remaining amount to. The sons discover that all their shares are equal. How many sons did the man have, and how large was his estate?
Various other problems are in the same vein; for example:
A man sneaks into an orchard to steal some apples. He has to leave via the guarded gates, of which there are seven to pass through. At the first gate, he gives the guard half the apples he is carrying and one extra, to keep quiet and let him through. At the next gate, he again gives the guard half of the apples he has and one extra. This continues for the remaining fives guards, half plus one each. The man finally gets out of the orchard and has one apple left. How many apples did he have to start with?

Problems involving progressions include:
A lion is in a 50ft well (the problem has been converted into modern units). Each day, he can climb up 1/7 of a foot, and each night he slips back 1/9 of a foot. How many days does it take him to climb out?
1/7 – 1/9 gives 2/63 feet up every 24 hours. 50/(2/63) = 1575, but the lion can get out as soon as he can reach the edge, so instead use
(50 – 1/7)/(2/63) = 1570½. Therefore after 1571 days, he is close enough to climb past the lip the next day, on day 1572.
The 'rabbit problem' that gives rise to the Fibonacci Sequence is also from *Liber Abaci*. For details of the problem, see the section entitled 'The Fibonacci Sequence'.
There are also simpler problems, many with a commercial connection:
A man has seven pounds of silver. He wishes to make coins with two ounces of silver per pound. How much alloy must he add to the silver for this?

Leonardo also gives examples of operations using the new Hindu numerals. I have included multiplication, and division by the 'scratch' method.
Multiplication of 934 and 314:

Start with

9	3	4		
				4
				1
				3

Multiplying 4 by 4 gives 16

9	3	4		
		1	6	4
				1
				3

3 × 4 = 12, and add the 1 from 16 to the 2

9	3	4		
	1	3	6	4
				1
				3

Continue for 4 × 9:

9	3	4		
3	7	3	6	4
				1
				3

Multiplying by 1 and 3 follow the same procedure, resulting in:

9	3	4		
3	7	3	6	4
	9	3	4	1
2	8	0	2	3

The numbers in the grid are added along the diagonal, with the unit digit of each total lining up with the start of each diagonal in the top row:
The unit of the solution is 6, then 3+4 = 7, 7+3+2 = 12, carrying 1 to the next total, 3+9+0(+1) = 13 carry the 1, 8(+1) = 9, and 2 = 2. Therefore the answer is 293276.

9	3	4		
3	7	3	6	4
	9	3	4	1
2	8	0	2	3

2 9 3 2 7 6

Leonardo investigates two types of division, the first case being division of a large number by a single digit number, which is done in the standard modern way. The second case is for two multi-digit numbers, for which Leonardo uses the 'scratch' method:
An example given is 65284 ÷ 594 = 109, remainder 538.

Begin with 65284)
594

5 goes into 6 once, with a remainder of 1:
Cross out the numbers that have been 'used'.

1 ←remainder
65284)1 ←quotient
594

Subtract the '9' in 594 from the '15' now evident in the dividend, putting the result, 6, next to the 1 and crossing out the numbers 'used':

16
65284)1
594

4 from 62 gives 58:

5
168
65284)1
594

Now re-write 594 underneath the first 594, shifted over one place:

5
168
65284)1
5944
59

594 is larger than 588, so a 0 is written in the quotient, and 594 is moved over again:

5
168
65284)10
59444
599
5

10 × 594 is larger than 5884, so work with 9:

```
 5
168
65284)109
59444
 599
  5
```

5 × 9 is 45, and 58 – 45 is 13:

```
 1
 53
168
65284)109
59444
 599
  5
```

138 – 9 × 9 = 57:

```
 15
 53
1687
65284)109
59444
 599
  5
```

74 – 4 × 9 = 38:

```
 15
 533
16878
65284)109
59444
 599
  5
```

Therefore the result is 109, with a remainder of 538, as expected.

Leonardo also included problems with a commercial basis:
Two men, A and B, have some denarii each. If B gives A 7 denarii, A will have five times the amount B has. If A gives B 5 denarii, B will have seven times A's amount. How much do A and B have?
I could not find Leonardo's solution, but my own calculations give:
A has 7 2/17 denarii, and B has 9 14/17 denarii.
Interest on an investment is also treated:
A man puts 1 denarius into an account. After 5 years, it has doubled to 2 denarii, and continues doubling every 5 years. How much will he have gained in 100 years?
Again, the details of the method used by Leonardo were not found, and my own calculations give a total of 2^{20} =1048576 denarii, a gain of 1048575.
Leonardo's strange method of writing fractions appears in another example found in *Liber Abaci*:
If $\frac{1\ 2}{4\ 3}$ of a rotulus is worth $\frac{1\ 1\ 2}{7\ 6\ 5}$ of a bizantium,

then $\frac{1\ 4\ \ 7}{8\ 9\ 10}$ of a bizantium is worth $\frac{3\ \ 8\ \ 83\ \ 11}{4\ 10\ 149\ 12}$ of a rotulus.[5]

In Leonardo's form, $\frac{1\ 4\ 7}{8\ 9\ 10}$ means $\frac{1}{8\times 9\times 10}+\frac{4}{9\times 10}+\frac{7}{10}$, but $\frac{1\ 2}{4\ 3}$ means $\frac{1}{4}+\frac{2}{3}$. This was one of the only flaws in Leonardo's method's.

Practica Geometriae

This book contains a proof of Hero's formula for the area of a triangle in terms of its sides a, b and c:
$A=\sqrt{s(s-a)(s-b)(s-c)}$, where $s=\frac{1}{2}(a+b+c)$
It also has a proof that the medians of a triangle divide each other in the ratio 2:1.

Flos

Compared to *Liber Abaci* and *Liber Quadratorum*, very little is known about the details of both *Flos* and *Practica Geometriae*. *Flos* concentrates on cubic equations.

Liber Quadratorum

Following are the details from all 24 propositions from the book.[13]

Proposition 1: 'Find two square numbers which sum to a square number.'
This is a simple method of generating Pythagorean triplets. It utilises the fact that the odd numbers from 1 to $(2n - 1)$ add to n^2, which is proven in Proposition 4. An odd square is chosen as the first square. Then every odd number less than this is added together, giving a square. As the next odd after this set is a square, adding it also produces a square, thereby completing the triplet.
For example, 9 is an odd square. Adding every odd number less than 9 gives 1 + 3 + 5 + 7 = 16, which is 4^2. Therefore, 16 + 9 = 1 + 3 + 5 + 7 + 9 = 25 = 5^2.

Proposition 2: 'Any square number exceeds the square immediately before it by the sum of the roots.'
This proposition examines $(n+1)^2 = n^2 + 2n + 1 = n^2 + (n+1) + n$.
Leonardo looks at the simple case described in the problem, and shows that if $(n+1) + n$ is a square, a Pythagorean triplet is created. Leonardo looks also at the variation $(n+1)^2 - (n-1)^2 = 4n$. He notes that if $4n$ is a square, and therefore n is a square, the result gives another method of generating a Pythagorean triplet. Finally, he shows that the difference

between any two squares is equal to the product of the sum of their roots and the (positive) difference of their roots, that is $a^2 – b^2 = (a + b)(a – b)$. The examples given in *Liber Quadratorum* are:
$11^2 = 121 = 100 + 21 = 10^2 + 11 + 10$.
If $n = 12$, $(n+1) + n = 13 + 12 = 25 = 5^2$. $\therefore$ $13^2 = 12^2 + 5^2$.
An example of the variation form given by Leonardo is:
If $n = 9$ (a square), $4n = 36$, also square. Then $n+1 = 10$, and $10^2 = 100$, and $n-1 = 8$, and $8^2 = 64$. $100 – 64 = 36$, giving the triplet $6^2 + 8^2 = 10^2$.

Proposition 3: 'There is another way of finding two squares which make a square number with their sum.'
In this proposition, Leonardo demonstrates yet another method of generating Pythagorean triplets, that he describes as obtaining from Book X of Euclid's *Elements*. He proves the form
$((a^2 + b^2)/2)^2 = ((a^2 – b^2)/2)^2 + (ab)^2$ by using another identity, that of dividing a line segment into equal and unequal parts. The formula is also widely known in the form $a^2 + b^2$, $a^2 – b^2$, $2ab$.

Proposition 4: 'I wish to demonstrate how a sequence of squares is produced from the ordered sums of odd numbers which run from one to infinity.'
This proves the important result that is used in various other Propositions in the book, that $1 + 3 + 5 + \ldots + (2n – 1) = n^2$. The argument is based on Proposition 2, the difference between two consecutive squares is the sum of their roots. Consecutive whole numbers are squared, and the differences are taken. These are in fact the odd numbers, except the final square is left. Summing all the differences gives the required result.
Numerically;

$$\begin{aligned} 1^2 &= 1 \\ 2^2 - 1^2 &= 3 \\ 3^2 - 2^2 &= 5 \\ 4^2 - 3^2 &= 7 \\ \vdots \quad & \quad \vdots \\ n^2 - (n-1)^2 &= 2n-1 \end{aligned}$$

Summing both sides gives $n^2 = 2n-1 + \ldots + 7 + 5 + 3 + 1$, as expected.

Proposition 5: 'Find two numbers so that the sum of their squares makes a square formed by the sum of the squares of two other given numbers.'
This is again Pythagorean triplets, this time finding two sets with the same hypotenuse. Leonardo uses similar triangles to form his solution, looking at three cases; whether the squared sum of the two given squares is equal to, greater, or less than the squared sum of the other two numbers. Modern algebra removes the need for three different cases, so

looking at one approach will be enough. If a and b are the given numbers, and $a^2 + b^2 = c^2$; take two other numbers x and y, and let $x^2 + y^2 = z^2$. Therefore, $(x/z)^2 + (y/z)^2 = 1$, and so multiplying both sides by c^2 gives $(xc/z)^2 + (yc/z)^2 = c^2$, as required.

Proposition 6: ‘A number is obtained which is equal to the sum of two squares in two, three, or four ways.’
This proposition uses the identities:

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (bc - ad)^2$$
$$= (ad + bc)^2 + (ac - bd)^2$$

also mentioned earlier. These give the first two sums of squares.
If $a^2 + b^2$ (or equally $c^2 + d^2$) is also a square, say e^2, a third solution is $e^2(c^2 + d^2) = (ec)^2 + (ed)^2$. If both $a^2 + b^2$ and $c^2 + d^2$ are square, the fourth solution, taking $c^2 + d^2 = f^2$, is $(a^2 + b^2)f^2 = (af)^2 + (bf)^2$.

Proposition 7: ‘Find in another way a square number which is equal to the sum of two square numbers.’
This is another variation on generating Pythagorean triplets, utilising Proposition 6. If we choose our values to set $bc - ad = 0$, then $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 = (ad + bc)^2 + (ac - bd)^2$, the sum of two squares. To achieve $bc - ad = 0$, the ratio a:c must equal b:d. This gives the three required values.
For example, if we have $a = 3$, $b = 4$, $c = 6$, $d = 8$; then

$$(3*6 + 4*8)^2 = (3*8 + 4*6)^2 + (4*8 - 3*6)^2$$
$$50^2 = 48^2 + 14^2$$

Proposition 8: ‘Two squares can again be found whose sum will be the square of the sum of the squares of any two given numbers.’
This again makes use of Proposition 6. By setting $c = a$ and $d = b$, $(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (bd - ac)^2$ becomes $(a^2 + b^2)^2 = (2ab)^2 + (b^2 - a^2)^2$. As a and b are the given numbers, the squares of the two numbers $2ab$ and $b^2 - a^2$ add to the square of the sum of the squares of the given numbers. This identity, for forming a Pythagorean triplet, was also arrived at in Proposition 3.

Proposition 9: ‘Find two numbers which have the sum of their squares equal to a nonsquare number which is itself the sum of the squares of two given numbers.’
This is essentially the same as Proposition 5, but the sum of the squares is not restricted to being a square itself. The argument is made in the same manner, using proportionality.
For example, if the two given numbers are 4 and 5, then $4^2 + 5^2 = 41$. Take two numbers whose squares add to a square, such as 3 and 4, giving

$3^2 + 4^2 = 5^2$. These four numbers become *a*, *b*, *c*, and *d* for the formula in Proposition 6. This gives $(3^2 + 4^2)(4^2 + 5^2) = 25*41 = 1025$.
Two squares which add to 1025 are: 3*4 + 4*5 = 32, and 4*4 – 3*5 = 1.
From this, $32^2 + 1^2 = 1025$. Dividing through by 25 gives
$(32/5)^2 + (1/5)^2 = 1025/25 = 41$, as required. Therefore, the answers are 6 2/5 and 1/5.

Proposition 10: Find the sum of the squares of consecutive numbers from the unity to the last.'
Leonardo begins by showing the following:
$n(n + 1)(2n + 1) = (n - 1)n(2n – 1) + 6n^2$,
and as $(n - 1)n(2n – 1) = (n - 1)((n - 1) + 1)(2(n - 1) + 1)$, the triple-product on the previous value, this provides a recursive-type relation for the rest of the proof. He then uses a listed form similar to that in Proposition 4, using the difference of the *n* and *n*-1 triple-products, to get the $6n^2$ on its own. The lists are added, and all the terms other than the largest cancel out, leaving the sum of the squares terms.

For example:
$$5(5+1)(2*5+1) – (5-1)5(2*5-1) = 6*5^2$$
$$4(4+1)(2*4+1) – (4-1)4(2*4-1) = 6*4^2$$
$$3(3+1)(2*3+1) – (3-1)3(2*3-1) = 6*3^2$$
$$2(2+1)(2*2+1) – (2-1)2(2*2-1) = 6*2^2$$
$$1(1+1)(2*1+1) = 6*1^2$$
Adding the rows gives: $5(5+1)(2*5+1) = 6(1^2+2^2+3^2+4^2+5^2)$,
5*6*11 = 330 = 6(1+4+9+16+25) = 6*55, as required.

Proposition 11: 'Find the sum of the squares of consecutive odd numbers from the unity to the last.'
The method used here is essentially the same as the previous proposition, but only involves odd numbers. In this case, with the final summed odd number (2*n*-1), the identity required is the triple-product of the odd number, the next odd number, and their sum;
$(2n-1)(2(n+1)-1)((2n-1)+(2(n+1)-1)) =$
$(2n-1)(2n+1)4n = (2(n-1)-1)(2(n-1)+1)4(n-1) + 12(2n-1)^2$
or $(2n-1)(2n+1)4n – (2n-3)(2n-1)4(n-1) = 12(2n-1)^2$
Working the same way as before, adding a list of equations so terms cancel, leaving the largest of the triple-products and the sum of the squared terms. For example (with $n = 4$):
$$7*9*16 – 5*7*12 = 12*7^2$$
$$5*7*12 – 3*5*8 = 12*5^2$$
$$3*5*8 – 1*3*4 = 12*3^2$$
$$1*3*4 = 12*1^2$$
Adding the rows and cancelling terms gives: $7*9*16 = 12(1^2+3^2+5^2+7^2)$
1008 = 12(1+9+25+49) = 12*84, as required.

Proposition 12: ‘If two numbers are relatively prime and have the same parity, then the product of the numbers and their sum and difference is a multiple of 24.’
Leonardo shows this result by looking at different combinations of the two numbers and their sum and difference, and how they relate to each other in two different cases. In having the same parity, Leonardo means that they are either both even or both odd; and as they are relatively prime, they must both be odd. From this, their sum and difference will both be even. The two cases considered are that half of the difference is odd, and half the difference is even. In the odd case, it is shown that half of the sum of the numbers will be even. Therefore, half the sum multiplied by half the difference will be even. As we have half and half, one quarter of the product of the sum and difference is even, or the sum-difference product is a multiple of 8. In the other case, half the difference is even. Therefore, half the sum multiplied by half the difference will be even, and again the sum-difference product will be a multiple of 8.
Leonardo then considers the numbers themselves. As the numbers are odd and relatively prime, either one only will be a multiple of three, or neither will. If one is, then the total product will be a multiple of three, from this number, and also a multiple of 8 from the sum-difference product; and is therefore a multiple of 24.
If neither number is a multiple of three, then on division by three they will have a remainder of either 1 or 2. If they both have the same remainder, the difference will then be a multiple of three, as the equal remainders will cancel out on subtraction. This gives the total a factor of three and a factor of 8 again, with the total product still a multiple of 24. If the remainders are different, adding the two numbers will result in an extra 3, so the sum will be a multiple of three, and the same result is evident. Leonardo uses the term ‘congruous’ for numbers of this form, namely $mn(n-m)(n+m)$.
Leonardo also mentions, but does not prove, the case where one of the two original numbers is even, when the formula he gives is $(2m)(2n)(n-m)(n+m)$. One of m or n is even, therefore the product $(2m)(2n)$ is a multiple of 8. The argument is the same as before to get the required factor of three, and the result stands.

Proposition 13: ‘The mean of symmetrically disposed numbers is the centre.’
Here, Leonardo considers numbers arranged about a central number, such that the difference between a larger and the centre is equal to the difference between the centre and a smaller number, and that the numbers only occur in these pairs. From this, it is easy to see that each pair adds to

twice the central number, and the total of all the numbers is equal to the central number multiplied by the number of numbers. If both sides are divided by the number of numbers, it is seen that the average is the result; that is the numbers added up and divided by the number of them.

Proposition 14: ‘Find a number which added to a square number and subtracted from a square number yields always a square number.’
Leonardo again makes use of the sum of odd numbers concept in the proof of this proposition. An alternative statement for the proposition is find a number which added to one square makes another square, and also added to that square makes a third square. From this, it can be seen that the required number must be simultaneously equal to two different sums of consecutive odd numbers, as all three squares are sums of consecutive odd numbers. The two sets of odd numbers are also adjacent, and the required number is congruous, as described above.
If the two equations are:

$$x^2 + c = y^2$$
$$y^2 + c = z^2$$

then c will comprise of y-x odd numbers, centred on y+x, giving $c = y^2 - x^2$ as before, using Proposition 13; and also z-y odd numbers centred on z+y, giving $c = z^2 - y^2$.
If we take two numbers, m, n, with n>m>0, to construct the solution; there are three cases Leonardo considers, involving the ratio of m to n and the ratio of (n-m) to (n+m). From these, there will be $n(n-m)$ odd numbers placed symmetrically about $m(n+m)$, and there will be $m(n-m)$ about $n(n+m)$. It can be seen that these come to the same total, $mn(n-m)(n+m)$. Using Propositions 4 and 13, the following result is obtained:

$$\{[m(n+m) - n(n-m)]/2\}^2 + mn(n-m)(n+m) = \{[m(n+m) + n(n-m)]/2\}^2$$
$$\{[m(n+m) + n(n-m)]/2\}^2 + mn(n-m)(n+m) = \{[n(n+m) + m(n-m)]/2\}^2$$

for the case m and n both odd or both even, with $m/n > (n-m)/(n+m)$. The other cases, the same parity but $m/n < (n-m)/(n+m)$, and with m and n of different parity, give very similar results.
For example, if m = 3 and n = 5, then n-m = 2 and n+m = 8. This gives: $n(n-m)$ = 10 odd numbers about $m(n+m)$ = 24, and $m(n-m)$ = 6 odd numbers about $n(n+m)$ = 40, and the total is $mn(n-m)(n+m)$ = 240.
The equations are therefore:

$$7^2 + 240 = 17^2$$
$$17^2 + 240 = 23^2$$

Proposition 15: ‘Square multiples of congruous numbers are congruous numbers.’
For the congruous number c: $x^2 + c = y^2$, $y^2 + c = z^2$. Therefore, if everything is multiplied by t^2, the equations become $(xt)^2 + ct^2 = (yt)^2$, $(yt)^2 + ct^2 = (zt)^2$, and so ct^2 is congruous for xt, yt, zt.

Proposition 16: ‘I wish to find a congruous number which is a square multiple of five.’
Leonardo chooses a congruous number of the form $4mn(n-m)(n+m)$ to be a multiple of 5. He puts $n = 5$, and takes m to be a square such that $m \pm n$ is also square, therefore $m = 4$. This makes the congruous number = 720, which is 144 * 5, a square multiple of 5.

Proposition 17: ‘I wish to find a square number which increased or diminished by five yields a square number.’
This is the question from the tournament for Frederick II, mentioned in the introduction to *Liber Quadratorum*. It is a specific case of Proposition 14, and it uses the result from the previous proposition.
Using the formulae developed in Proposition 14, the following equations are obtained, with $m = 4$ and $n = 5$:

$$n+m = 9 \qquad n-m = 1$$

$$n/m = 5/4 < 9/1 = (n+m)/(n-m) \text{ - testing for which form to use}$$

$$\therefore\ 2n(n-m) = 10 \qquad 2m(n-m) = 8$$

$$2n(n+m) = 90 \qquad 2m(n+m) = 72$$

$$(1/2)(72 - 10) = 31$$

$$(1/2)(72 + 10) = 41$$

$$(1/2)(90 + 8) = 49$$

$$\therefore\ 31^2 + 720 = 41^2 \text{ and } 41^2 + 720 = 49^2$$

Dividing through by 144 gives the required result:

$$(31/12)^2 + 5 = (41/12)^2 \text{ and } (41/12)^2 + 5 = (49/12)^2$$

Proposition 18: ‘If any two numbers have an even sum, then the ratio of their sum to their difference will not be the same as the ratio of the larger to the smaller.’
This means to show that $n/m \neq (n+m)/(n-m)$ for $n > m > 0$
If we suppose that $n/m = (n+m)/(n-m)$, then $n(n-m) = m(n+m)$ or
$n^2 - nm = mn + m^2 \rightarrow m^2 + 2mn = n^2$: add n^2 to both sides
$\rightarrow m^2 + 2mn + n^2 = 2n^2 \rightarrow (m+n)^2 = 2n^2$
But the ratio of two whole squares cannot equal 2, giving a contradiction, and therefore the initial statement must be true.
Leonardo uses a slightly different method to obtain the same result, utilising the familiar sum of odd numbers theory.

Proposition 19: 'Find a square number for which the sum and difference of it and its root is a square number.'
The solution of this comes from the congruous number equations:
$y^2 - c = x^2$ and $y^2 + c = z^2$
Dividing through by c gives:
$y^2/c – 1 = x^2/c$ and $y^2/c + 1 = z^2/c$
Multiplying by y^2/c, we obtain the result:
$(y^2/c)(y^2/c) – (y^2/c) = (x^2/c)(y^2/c)$ and
$(y^2/c)(y^2/c) + (y^2/c) = (z^2/c)(y^2/c)$
or
$(y^2/c)^2 – (y^2/c) = (xy/c)^2$ and $(y^2/c)^2 + (y^2/c) = (yz/c)^2$
as required.
For example:

25 – 24 = 1	25 + 24 = 49
25/24 – 1 = 1/24	25/24 + 1 = 49/24
$(25/24)^2 – (25/24) = (5/24)^2$	$(25/24)^2 + (25/24) = (35/24)^2$

Proposition 20: 'Similarly, a square number must be found which when twice its root is added or subtracted always makes a square number.'
This is essentially the same as the previous proposition, but the equations are multiplied by two when the '1' is evident.
(from above)
$y^2/c – 1 = x^2/c$ and $y^2/c + 1 = z^2/c$ multiply by 2
$2y^2/c – 2 = 2x^2/c$ and $2y^2/c + 2 = 2z^2/c$
We now multiply both by $2y^2/c$:
$(2y^2/c)(2y^2/c) – 2(2y^2/c) = (2x^2/c)(2y^2/c)$ and
$(2y^2/c)(2y^2/c) + 2(2y^2/c) = (2y^2/c)(2z^2/c)$
or
$(2y^2/c)^2 – 2(2y^2/c) = (2xy/c)^2$ and $(2y^2/c)^2 + 2(2y^2/c) = (2yz/c)^2$

Proposition 21: 'For any three consecutive odd squares, the greatest square exceeds the middle square by eight more than the middle square exceeds the least square.'
If the three consecutive odd squares are $(2n+1)^2$, $(2n+3)^2$ and $(2n+5)^2$, then the difference between the pairs is:
$(2n+5)^2 - (2n+3)^2 = 2*(4n+8)$ and $(2n+3)^2 - (2n+1)^2 = 2*(4n+4)$
and the difference between the differences is
$(8n+16) – (8n+8) = 8$, as said in the beginning.
This gives a principle for generating odd squares:
$1^2 = 1$
$3^2 = 9 = 1 + 8$
$5^2 = 25 = 1 + 8 + 2*8$
$7^2 = 49 = 1 + 8 + 2*8 + 3*8$

Proposition 22: ‘I wish to find in a given ratio the two differences between three squares.’
Expressed in modern notation, this is:
$x^2 + ta = y^2$ and $y^2 + tb = z^2$
or $(y^2 – x^2)/(z^2 – y^2) = ta/tb = a/b$, the given ratio.
If $a = b$, that is the ratio is 1, this becomes the same problem as Proposition 14.
Leonardo uses Propositions 4 and 21 in his proof of this proposition, which is also found in Book II of Diophantus’ *Arithmetica*.
Leonardo considers various cases for the values a and b, beginning with $b = a + 1$. As shown previously, consecutive odd squares differ by a multiple of 8, and so, taking the three odd squares a, a+1, a+2:
$(2(a+1)-1)^2 – (2a-1)^2 = (1+8+…+8a) – (1+8+…+8(a-1)) = 8a$, and
$(2(a+2)-1)^2 – (2(a+1)-1)^2 = (1+8+…+8(a+1)) – (1+8+…+8a) = 8(a+1)$
and the ratio of the differences is $8a/8(a+1) = a/(a+1) = a/b$ as required.
The next case treated is when a and b are consecutive odd numbers. The formula required is an adaptation of Proposition 21:
$(2p)^2 = 4 + 3*4 + 5*4 + … + (2p-1)*4$, summing to form an even square.
With the differences acquired in the same manner as before from the squares of $2(t-1)$, $2t$ and $2(t+1)$; the ratio becomes $4(2t-1)/4(2t+1) = (2t-1)/(2t+1) = a/b$.
Leonardo also gives an example where a and b are squares, namely a = 16 and b = 25. Taking the geometric mean of these gives c = 20; therefore the three squares are $16^2 = 256$, $20^2 = 400$ and $25^2 = 625$. 400 – 256 = 144, and 625 – 400 = 225, and 144/225 = 16/25 = a/b.
For problems that do not fit into any of the previous cases, Leonardo gives a vague method, outlined by another example:
If a/b = 2/9, then find a ‘suitable’ ratio of multiples of 8 in the multiples of the ratio, that is 2/9, 4/18, 6/27, … In 4/18, Leonardo finds his requirements, as 18 = 5+6+7, and the solution stems from 4/18 = (4*8)/(5*8+6*8+7*8).
$(4*8+3*8+2*8+8+1) – (3*8+2*8+8+1) = 9^2 – 7^2 = 4*8$, and
$(7*8+6*8+…+8+1) – (4*8+3*8+2*8+8+1) = 15^2 - 9^2 = 8*(7+6+5)$, so giving the solution.

Proposition 23: ‘I wish to find three squares so that the sum of the first and the second as well as all three numbers are square numbers.’
This follows easily from Proposition 4. Beginning with two squares that add to a square, another square is found by adding up all the odd numbers less than this sum square. This gives a square, which added to the sum square, gives a square as well, as a sequence of odd numbers is still evident. For example, 9 + 16 = 25. Adding all odd numbers less than 25

gives 144, a square, and adding 25 to this gives 169, also a square. A formula can also be used generate the next square; if a is a square, then $a + ((a-1)/2)^2 = ((a+1)/2)^2$, and from the example, $a = 25$, $(a-1)/2 = 12$, and $(a+1)/2 = 13$, the same results as before.

Proposition 24: ‘I wish to find three numbers which added together with the square of the first number make a square number. Moreover, this square, if added to the square of the second number, yields thence a square number. To this square, if the square of the third number is added, a square number similarly results.’
This problem was given to Leonardo by Theodorus, a philosopher in Frederick’s court. Put more simply, we want x, y, z, such that $x + y + z + x^2$ is a square, $x + y + z + x^2 + y^2$ is a square, and $x + y + z + x^2 + y^2 + z^2$ is also a square. The solution utilises Proposition 23 to find a set of squares to begin working with. Leonardo uses the following equations:
$6^2 + 8^2 = 10^2$ and $10^2 + 24^2 = 26^2$. Then he works to find a number x so that: $x + 8 + 24 + x^2 = 6^2$; and then this with $6^2 + 8^2 = 10^2$ and $10^2 + 24^2 = 26^2$ gives the solution to the problem. Simplified, x is the solution to $x^2 + x + ¼ = 17/4$, and $x = ½(-1+\sqrt{17})$. Leonardo then has to find a rational solution to the problem, by looking at multiples of the whole numbers he started with, namely 6, 8, 10, 24, and 26; that is he needs to find rational solutions to: $x + 8k + 24k + x^2 = (6k)^2$, $(6k)^2 + (8k)^2 = (10k)^2$, $(10k)^2 + (24k)^2 = (26k)^2$. Substituting $R=6k$ into the first equation gives $x^2 + x = R^2 - (16/3)R$. With another substitution of $x = R – a$, the equation becomes $(R – a)(R – a + 1) = R^2 - (16/3)R$. Solving for R gives
$R = 3a(a-1)/(6a-19)$, and so $x = a(3a-16)/(19-6a)$. For $19/6 < a < 16/3$, x and R will be positive. Leonardo chooses $a = 4$, giving $x = 16/5$, $y = 48/5$, and $z = 144/5$, a solution to the problem. Leonardo also calculates an integer solution following the same method with different starting values. It is $x = 35$, $y = 144$, and $z =360$. He goes on to generalize the formula for more variables, and gives a solution for four unknowns, $w = 1295$, $x = 31968/7$, $y = 79920/7$, and $z = 79920$.

The Fibonacci Sequence

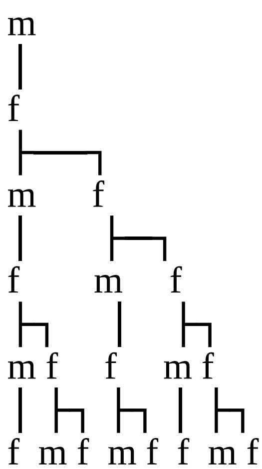
The Fibonacci Sequence gains its name from its appearance in a problem from *Liber Abaci*:
A young pair of rabbits is put in an enclosure. They produce one pair of offspring each month, and the offspring cannot breed until their second month, when they too produce one pair of offspring per month. Assuming no mortality, how many rabbits would there be in total after one year?
This begins the sequence, and at the end of the first month, there is still only one pair. They reproduce, so at the end of the second month there

are two pairs. At the end of the third month, the original pair have reproduced again, but the new pair have not, making three pairs. At the end of the fourth month, both of the first two pairs have reproduced, but again the newest pair have not, giving five pairs. In each subsequent month, all those alive two months ago have added another pair to the number that were alive last month. This provides the well-known formula:

$$f_n = f_{n-1} + f_{n-2}$$

There are various other ways of describing the problem with the same numerical outcome. If instead of counting all the rabbits at each month, we only count the number produced, and the rabbits only produce a pair for two generations, the same sequence is obtained. It also comes about in the study of the genealogy of the bee. A male bee has only a mother, but a female bee has a mother and a father. Working backwards in time in this case, one male bee has, from the previous generation, one female parent. She, on the other hand, has both one male and one female predecessor, making two relatives. This male has only one mother, and the female has one mother and one father, making three in this generation. Consider the following diagram:

Generation	Males	Females	Total
n	1	0	1
n-1	0	1	1
n-2	1	1	2
n-3	1	2	3
n-4	2	3	5
n-5	3	5	8



As can be seen, all three totals form a Fibonacci sequence, going back through the generations. Note that zero is sometimes included in the sequence as the ‘zero-th’ term.[12]

There are many interesting properties of the sequence, involving terms that are consecutive or nearly consecutive. For example:

$$f_{n+1}.f_{n-1} = f_n^2 + (-1)^n$$

$$f_{m+n} = f_{n-1}.f_m + f_n.f_{m+1}$$

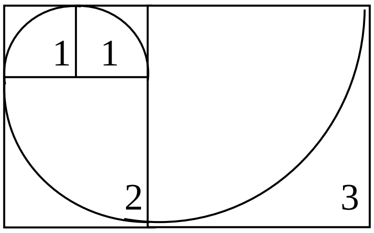
$$\mathrm{GCD}(f_n, f_{n+1}) = 1 \text{ (that is, relatively prime)}$$

$$f_1 + f_2 + \dots + f_n = f_{n+2} - 1$$

for the sum of the first n Fibonacci numbers. Also the sum of the squares of the first n Fibonacci numbers:

$$f_1^2 + f_2^2 + \dots + f_n^2 = f_n . f_{n+1}$$

a formula that arises from fitting the squares together to form a rectangle. As the side of each square is the sum of the sides of the two before it, a rectangle results:



Another intriguing property of this square layout is that a logarithmic spiral will fit perfectly through common vertex of each consecutive pair. (My diagram is a circular-arc approximation).
There is an exact formula for any Fibonacci number:

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$\alpha = \tfrac{1+\sqrt{5}}{2}, \beta = \tfrac{1-\sqrt{5}}{2}$$

This is known as Binet's Formula, after the Frenchman Jacques Binet, who worked it out in 1843. It was also known to Leonhard Euler and Daniel Bernoulli a century earlier.
The formula develops from the fact that the ratio of consecutive Fibonacci numbers approaches the Golden Ratio:

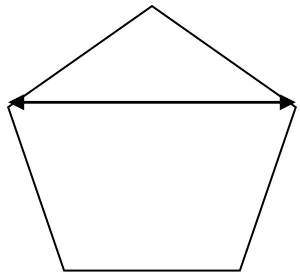
$$\lim_{n\to\infty}\left(\frac{f_{n+1}}{f_n}\right) = \frac{1+\sqrt{5}}{2} \approx 1.6180339\ldots$$

The Golden Ratio, ϕ, itself has some interesting features:

$$\phi^2 = \phi + 1$$

$$1/\phi = \phi - 1$$

The Golden Ratio is also found in a regular pentagon. If the pentagon has sides of length 1, then the width will be 1.6180339… Related to this is



the fact that an isosceles triangle with a base angle of 72º has a side-to-base ratio of 1.6180339… Because of this, 72º is sometimes known as the "Golden Angle".

Some of the formulae relating to the Fibonacci Sequence can be derived from a matrix form:

$$\text{If } Q = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\text{then } Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

To prove this, first check that the first step works:

$$Q^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{Now assume } Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

$$\text{then } Q^{n+1} = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f_{n+1} + f_n & f_n \\ f_n + f_{n-1} & f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix}$$

and the inductive proof works. Many more relations between various Fibonacci numbers can be obtained from the matrix form, such as:

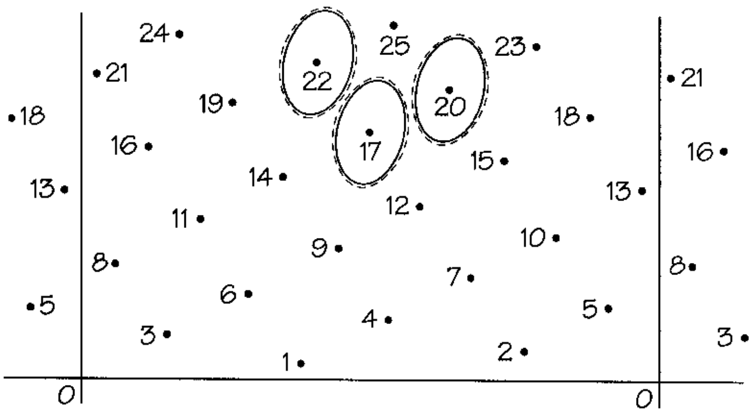
$$f_{n+1}^2 + f_n^2 = f_{2n+1}$$

$$\text{and } f_{m+n} = f_{n-1} f_m + f_n f_{m+1}$$

The Fibonacci Sequence can also be found in Pascal's Triangle. If the triangle is drawn with the left side vertical, the 45° totals will be the Fibonacci Sequence:

		1								
	⋰		1							
1		⋰		2						
	⋰		⋰		3					
1		1		⋰		5				
	⋰		⋰		⋰		8			
1		2		1		⋰		⋱		
	⋰		⋰		⋰		⋰		⋱	
1		3		3		1		⋰		⋱
	⋰		⋰		⋰		⋰		⋰	
1		4		6		4		1		⋰
	⋰		⋰		⋰		⋰		⋰	
1		5		10		10		5		1

One of the most interesting occurrences of the Fibonacci Sequence is in nature. Because of the way leaves and petals are generated, they often form a Fibonacci number. The seeds on the head of a sunflower form two sets of spirals, one clockwise and one anti-clockwise. There are normally 34 spirals in one direction and 55 in the other. The 'cells' on a pinecone or a pineapple are also two consecutive Fibonacci numbers. Leaves along a stem also display similar behaviour. As the shoot or flower grows, the buds appear around the tip one at a time. Each successive bud grows by default into the largest gap among the previous buds. If the first two buds are not exactly opposite each other, the bud sequence will be distributed around the tip in the multiple-spiral pattern as the shoot grows. In the 'unwrapped' shoot diagram,



the bud numbered 25 grows in the gap among 17, 20 and 22, continuing the steep spiral through 5, 10, 15 and 20, a difference of 5; and the shallow spiral though 16, 19 and 22, a difference of 3. 0 is the position of the first bud, at 0º. The Fibonacci pattern is more noticeable for small numbers, as too much randomness can be evident with larger numbers of buds.[7]

There are some unanswered theoretical questions regarding the sequence, such as how many terms are prime numbers? There are 12 in the first 100 terms, but it is not known how they continue. Also, there are only two squares in the first 100 terms, 1 and 144 (f_1, f_2, and f_{12}).

Biographies

I include small biographies of the people mentioned in this essay.

Abû Kâmil[5]
A textbook writer living in Egypt, his full name was Abû-Kâmil Shoja ben Aslam. He was born in 850 and died in 930.

Al-Khowârizmî[20]
Al-Khowârizmî was born around 780AD. His full name was Abū Ja-far Muhammad ibn Mū sā al-Khwā-rizmī. He became a member of 'Dār al-Hikma', (House of Wisdom) in Baghdad. He worked on astronomy, mathematics, made maps of the Mediterranean and the Near East, and made an astrolabe, sundials and calendars. He wrote a practical mathematics handbook entitled 'Al-Kitāb al-mukhtasar fīhisāb al-jabr wa'l muqābala', which used 'shay' (meaning 'thing') as a variable and described Hindu numerals and operations with them. He died around 850AD. A corruption of his name gives us the term algorithm, and the word algebra comes from 'al-jabr' in the title of his work.

Daniel Bernoulli[20]
This Bernoulli was born in Gröningen, the Netherlands, in 1700; the second son of Johann Bernoulli. He studied logic, philosophy and mathematics, gaining a Masters degree at 16. He went on to study medicine, finishing that course of study in 1721. Unable to get a job in Switzerland, he went to Italy to continue studying medicine and mathematics. In 1725, he went to the St. Petersberg Academy of Sciences for a teaching position. There, he worked on muscular contraction, the optic nerve, oscillations, the parallelogram of forces, and probability. He later took a position in botany in Basel, just to get back to Switzerland. In 1737, he made calculations of the work done by the heart, and in 1738 wrote 'Hydrodynamica' on fluids, noting that pressure decreases as fluid velocity increases, and that pressure is proportional to temperature. He was a popular lecturer, and became a professor of physiology in 1743. He went on to gain the chair of Natural Philosophy in 1750. He studied the conservation of energy and acoustics, finding mathematical descriptions of sound waveforms and natural frequencies of musical instruments. Bernoulli died on March 17, 1782 in Basel.

Jacques P. M. Binet[4]
Binet was born in 1786, and lived in France. He is known to have worked on the matrix determinant around 1812, formulating a multiplication rule

in relation to solving simultaneous equations. He also worked with the explicit formula for the Fibonacci Sequence, in 1843. He died in 1856.

Baldassarre Boncompagni[13]
Boncompagni was a 19th century scholar in Italy, who edited and published the complete works of Leonardo after finding them in the Ambrosian Library in Milan, in two volumes entitled ‘Scritti di Leonardo Pisano’.

Diophantus[20]
Diophantus was alive around 270AD in Greece, although he may have been born in Alexandria. He worked on linear mathematics and early algebra, especially what are now known as Diophantine Equations. He wrote ‘Arithmetica’, on abstract arithmetic and early forms of algebra, and was known to have used a symbol for an unknown quantity in equations.

Dominicus[6]
An astronomer to Frederick II. Dominicus introduced Leonardo to the Emperor’s court before the tournament.

Euclid of Alexandria[20]
Euclid was alive around 300BC, and he may have attended Plato’s Academy in Athens. He taught at Ptolemy’s Museum/Library, and was known as a good teacher and a nice person. His main work was the ‘Elements’, which drew on Hippocrates, Pythagoras, and Menaechmus. It consisted of 13 books, containing around 450 propositions. It was both an accumulation of previous mathematicians work and Euclid’s own original concepts. The ‘Elements’ was used by later scientists such as Galileo, Newton, Archimedes and Eratosthenes. He also wrote ‘Data’ on plane geometry, ‘Phenomena’ on spherical geometry, ‘Optics’ on perspective and related subjects, ‘Pseudaria’ which included mathematical fallacies of the day and also valid theorems contrary to them, ‘Porisms’ on higher geometry, ‘Conics’, and ‘Surface-Loci’ on points and surfaces.

Leonhard Euler[20]
Euler was born on April 17, 1707, in Basel, Switzerland. His father was a pastor and also a mathematician, who wanted Euler to inherit his pastorship. Euler studied religion and mathematics, gaining a Masters degree at 17. He started losing his sight at an early stage. Euler was taught by the Bernoullis, and they wanted him to concentrate on maths. In 1727, Euler went to the St. Petersberg Academy of Sciences for a medical position. He soon moved to the mathematics department, joining Daniel

Bernoulli. When Bernoulli left, Euler assumed the top position. Euler covered many fields, including analytic geometry, differential and integral calculus, spherical trigonometry, algebraic series, number theory, hydrodynamics, graph theory (after working on the 'Konigsberg Bridges' problem in 1736), and also worked on geography and a system of weights and measures for the Russian government. He moved to Berlin in 1740 after an invitation to the Berlin Academy from Frederick the Great. There he worked on pension plans, navigation and coinage for the government, and published his work on 'Calculus of Variations' in 1746. He was made a Fellow of the Royal Society of London, and made some early progress on Fermat's Last Theorem. In 1766, he returned to Russia, and soon lost his eyesight completely. He kept on working, having a remarkable ability at mental arithmetic, looking at lunar theory, phases and tides. He wrote a treatise on integral calculus in 1768-70. Euler married twice, marrying his first wife's half-sister (who was also her aunt) after the first wife's death. Only a few of his many children survived their early years. Euler died on September 8, 1783, soon after calculating the orbit of the newly found Uranus.

Frederick II[2]

Frederick was born on December 26, 1194, in Jesi, Ancona, Italy. He was a member of the House of Hohenstaufen, the son of Henry VI and Constance, the heiress of the Two Sicilies. He was left an orphan in 1198, and brought up under wardship to the Pope. In 1208, he was in the government of the Two Sicilies, and in 1212 was made Aspirant to the Crown of Germany by the Pope as opposition to King Otto IV. He was duly elected by the Ghibelline Party, and crowned as King of Germany at Aachen in 1215. He was then crowned Holy Roman Emperor in 1220 by Pope Honorius III. He tried to combine Italy and Germany into one country. Between 1228 and 1229, he led a crusade to the holy land, and recovered Jerusalem by treaty and without fighting. He maintained his court at Naples, and surrounded himself with many scholars; he also founded the University of Naples. He reportedly spoke Latin, Italian, German, French, Greek and Arabic, and wrote a book on falconry and the care of falcons. He was hot tempered and ruled with an iron fist, but in a fair and well-supported manner. He argued with the Pope, and was excommunicated three times. He centralised authority in the monarch and made legislative reforms while he was in power. Frederick was Emperor until his death on December 13, 1250 in Fiorentino.

Hero (or Heron) of Alexandria[20]

Hero was alive around 65-125AD, and was a prolific writer and scholar, particularly concerning geometry and engineering. His books included

‘Metrica’ about geometry, areas and volumes, containing Hero’s formula for the area of a triangle; ‘Definitions’, a catalogue of geometry terms; ‘Geometrica’, an introduction to geometry; ‘Sterometrica’, about solid geometry such as spheres and pyramids; ‘Pneumatica’, discussing devices powered by steam and compressed air, such as a compressed-air powered catapult; ‘Mechanica’, on levers, pulleys and compound pulleys, screws and designs for mechanically-powered religious ‘miracles’; ‘Automata’, machines including mechanical puppets; ‘Dioptra’, about surveying and the dioptra, an instrument similar to a theodolite; ‘Catoptrica’, on mirrors and refraction; and ‘Baroulkos’, a lost volume on lifting heavy objects, presumably with mechanical assistance. He was interested in the practical applications of science, designing and building the Aeolipile, a steam powered rotor device,[14] predating the modern steam turbine by nearly two thousand years. He also worked on catapults; time keeping including a water clock, vault construction, and coin-operated machines.

Johannes of Palermo[16]
Court scholar to Frederick II, who set the questions in the tournament for Leonardo.

Guillaume Libri[8]
A 19th century historian, who may have been the first to give the name ‘Fibonacci’ to Leonardo Pisano. Libri was born in 1803, and died in 1869.

Blaise Pascal[20]
Pascal was born on June 19, 1623, in Clermont, Auvergne, France. His mother, Antoinette, died when he was three. Pascal was taught at home, mostly by his father, Ètienne. In 1631, the family moved to Paris, where Pascal started his formal education in mathematics and ancient languages. In 1639, his father got a job in the tax office in Rouen, and the family moved again. In 1640, at only 16, Pascal wrote a treatise on projective geometry and conics, containing over 400 propositions, including ‘Pascals mystic hexagram’, that showed how the three points of intersection of the extended opposite sides of a hexagon inscribed in a conic section lie in a straight line. He developed a mechanical calculator to help his fathers tax work, worked on atmospheric and barometric pressure, vacuums, invented the syringe and the hydraulic press, discovered that pressure is transmitted equally to all of a fluid, and looked at hydrostatics. In 1646, he became a Jansenist (anti Jesuit Catholic), and moved back to Paris in 1647. He corresponded with Pierre de Fermat in 1654 on probability and games of chance, and studied the arithmetic triangle that now bears his name, in which each entry is the sum of the

two entries above it, during work on binomial expansions. This laid much of the groundwork for Isaac Newton to formulate the general binomial theorem. He also studied the cycloid, and found the centre of area of a segment of cycloid, and the volume and surface area of a revolved cycloidal segment. Pascal became very sick in 1658, and died on August 19, 1662 at his sisters home.

Plato of Tivoli[9]
Translated many works into Latin in the 12th century, including an encyclopaedia by Savasorda, works on astronomy by al-Battânî, and work on spherics by Theodosius.

Savasorda[11]
Also known as Abraham bar Hiyya ha-Nasi. Savasorda may have been a co-worker of Plato of Tivoli. He wrote on arithmetic, geometry, optics and music. He was Jewish, but lived in Spain; and died in 1136.

Michael Scot[16]
Scottish astronomer and astrologer to Frederick II. Leonardo dedicated the second edition of *Liber Abaci* to him.

Theodorus[20]
A philosopher in the court of Frederick II. Leonardo corresponded with him, and one letter has survived, called *Epsistola ad Magistrum Theodorum*.

Bibliography

1 David Abbott (Ed.), ‘Mathematicians’, Muller, Blond and White Ltd., 1985

2 Catherine B. Avery (Ed.), ‘The New Century Italian Renaissance Encyclopaedia’, Meredith Corporation, 1972

3 W. W. Rouse Ball, ‘A Short Account of the History of Mathematics’, MacMillan and Co. Ltd., 1924

4 Eric Temple Bell, ‘The Development of Mathematics’, McGraw-Hill Book Company, Inc., 1945

5 Carl B. Boyer, Uta C. Merzbach, ‘A History of Mathematics’, John Wiley and Sons, Inc., 1989

6 Florian Cajori, ‘A History of Mathematics’, The MacMillan Company, 1922

7 John H. Conway, Richard K. Guy, ‘The Book of Numbers’, Springer-Verlag New York Inc., 1996

8 Roger Cooke, ‘The History of Mathematics: A Brief Course’, John Wiley and Sons, Inc., 1992

9 Howard Eves, ‘An Introduction to the History of Mathematics’, Holt, Rinehart and Winston, 1964

10 Howard Eves, ‘Great Moments in Mathematics Before 1650’, The Mathematical Association of America (Inc.), 1983

11 Ivor Grattan-Guinness, ‘The Norton History of the Mathematical Sciences’, W. W. Norton and Company Inc., 1998

12 Ross Honsberger, ‘Mathematical Gems III’, Mathematical Association of America, 1985

13 Leonardo Pisano, ‘The Book of Squares’, translated by L. E. Sigler, Academic Press Inc., 1987

14 Struan Reid, ‘Invention and Discovery’, Usborne Publishing Ltd., 1986

15 Paul Lawrence Rose, ‘The Italian Renaissance of Mathematics’, Librarie Droz Genève, 1976

16 Vera Sandford, ‘A Short History of Mathematics’, George C. Harrap and Company Ltd., 1930

17 Joseph Frederick Scott, ‘A History of Mathematics’, Taylor and Francis Ltd., 1969

18 Michael Upshall (Ed.), ‘The Hutchinson Paperback Dictionary of Biography’, Arrow Books Ltd, 1990

19 Lawrence Young, ‘Mathematicians and Their Times’, North-Holland Publishing Company, 1981

20 Robyn V. Young (Ed.), ‘Notable Mathematicians’, Gale Research, 1997

21 ‘Microsoft Encarta Interactive World Atlas 2000’, Microsoft Corporation, 1999 (for the map)